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Rayleigh-Lamb Waves in Transversely Isotropic Thermoelastic Diffusive Layer

Rajneesh Kumar · Tarun Kansal

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Abstract Propagation of plane harmonic thermoelastic diffusive waves in a homogeneous, transversely isotropic, thin elastic layer of finite width is studied, in the context of the theory of coupled thermoelastic diffusion. According to the characteristic equation, three quasi-longitudinal waves, namely, quasi-elastodiffusive (QED) mode, quasi-mass diffusion (QMD) mode, and quasi-thermodiffusive (QTD) mode can propagate in addition to quasi-transverse waves (QSV) mode and the purely quasi-transverse motion (QSH) mode, which is not affected by thermal and diffusion vibrations, gets decoupled from the rest of the motion of wave propagation. The secular equations corresponding to the symmetric and skew symmetric modes of the layer are derived. The amplitudes of displacements, temperature change, and concentration for symmetric and skew symmetric modes of vibration of the layer are computed numerically. Anisotropy and diffusion effects on the phase velocity, attenuation coefficient, and amplitudes of displacements, temperature change, and concentration are presented graphically in order to illustrate and compare the results analytically. Some special cases of the frequency equation are also deduced and compared with the existing results.

Keywords Amplitudes · Attenuation coefficient · Coupled thermoelastic diffusion · Phase velocity · Transverse isotropy · Wave propagation

R. Kumar (✉) · T. Kansal
Department of Mathematics,
Kurukshetra University, Kurukshetra 136119, India
e-mail: rajneesh_kuk@rediffmail.com

T. Kansal
e-mail: tarun1_kansal@yahoo.co.in

1 Introduction

Thermoelasticity deals with the study of thermodynamic systems of bodies in equilibrium, whose interaction with the surroundings is limited to mechanical work, heat exchange, and external work. In general, a change of the body temperature is caused not only by external and internal heat sources, but also by the process of deformation itself. Under normal conditions of heat exchange, the flux produced by the deformation gives rise to unsteady heating. In classical theory, this change of temperature is very small. The corresponding terms in the field equations, inertia terms in the elastic equations of motion, and coupling term in the heat conduction equation are neglected, and the problem is treated as quasi-static. However, this is not true if the temperature undergoes a large and sudden change such as sudden heating or cooling of a body. In such cases, the inertia term must be considered in the equations of motion.

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain an elastic term; second, the heat conduction equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

The theory of coupling of thermal and strain fields gives rise to the coupled theory of thermoelasticity and was first postulated by Duhamel [1] shortly after the formation of the theory of elasticity. He derived the equations for the distribution of strains in an elastic medium subjected to a temperature gradient and introduced the dilatation term in the heat conduction equation, but this equation was not on a thermodynamic basis. Neuman [2], Voigt [3], and Jeffreys [4] made attempts at thermodynamical justification of the equations of Duhamel's theory and solved a number of interesting problems. The work of Biot [5] gave a satisfactory derivation of the heat conduction equation, which includes the dilatation term based on thermodynamics of irreversible processes.

Diffusion can be defined as the random walk of an assembly of particles from regions of high concentration to those of low concentration. Nowadays, there is a great deal of interest in the study of this phenomenon due to its application in geophysics and the electronics industry. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substance. In particular, diffusion is used to form the base and emitter in bipolar transistors, integrated resistors, and the source/drain regions in metal oxide semiconductor (MOS) transistors and doped poly-silicon gates in MOS transistors. In most of the applications, the concentration is calculated using what is known as Fick's law. This is a simple law which does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of temperature on this interaction. The study of the phenomenon of diffusion is used to improve the conditions of oil extraction (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits.

Until recently, thermodiffusion in solids, especially in metals, was considered as a quantity that is independent of body deformation. Practice, however, indicates that the process of thermodiffusion could have a very considerable influence upon the deformation of the body.

The thermodiffusion in elastic solids is due to the coupling of fields of temperature, mass diffusion, and that of strain in addition to heat and mass exchange with an environment. Nowacki [6–9] developed the theory of thermoelastic diffusion by using a coupled thermoelastic model. Dudziak and Kowalski [10] and Olesiak and Pyryev [11], respectively, discussed the theory of thermodiffusion and coupled quasi-stationary problems of thermal diffusion for an elastic layer. They studied the influence of cross effects arising from the coupling of the fields of temperature, mass diffusion, and strain due to which the thermal excitation results in additional mass concentration and which generates additional fields of temperature. Sherief et al. [12] developed the generalized theory of thermoelastic diffusion with one relaxation time, which allows finite speeds of propagation of waves. Aouadi [13–17] investigated different types of problems in thrmoelastic diffusion. Sharma et al. [18,19] discussed plane harmonic generalized thermoelastic diffusive waves and elasto-thermodiffusive surface waves in heat conducting solids.

Sharma [20] discussed the propagation of thermoelastic waves in homogeneous isotropic plates. Sharma and Pathania [21,22] investigated the generalized thermoelastic Lamb waves in a plate with layers of an inviscid liquid. Sharma and Pathania [23] discussed the generalized thermoelstic waves in anisotropic plates sandwiched between liquid layers.

The present investigation is concerned with the propagation of waves in a homogeneous, transversely isotropic (TI), coupled thermoelastic diffusive layer. The phase velocities and attenuation coefficients of various possible modes of wave propagation have been computed using the irreducible case of Cardano's method with the help of DeMoivre's theorem from the secular equations. The analytical results have also been computed numerically and represented graphically for illustration of various physical phenomena exhibited by such solids.

2 Basic Equations

The basic governing equations for an anisotropic, coupled thermoelastic difffusive solid in the absence of body forces, heat sources, and diffusive mass sources are given by

$$\sigma_{ij} = c_{ijkm}e_{km} + a_{ij}T + b_{ij}C, \tag{1}$$

$$q_i = -K_{ij}T_{,j}, \tag{2}$$

$$\eta_i = -\alpha^*_{ij}P_{,j}, \tag{3}$$

$$\rho ST_0 = \rho C_E T + aT_0C - a_{ij}e_{ij}T_0, \tag{4}$$

$$P = b_{km}e_{km} + bC - aT. \tag{5}$$

Here $c_{ijkm}(c_{ijkm} = c_{kmij} = c_{jikm} = c_{ijmk})$ are elastic parameters. $a_{ij}(= a_{ji}), b_{ij}$ $(= b_{ji})$ are tensors of thermal and diffusion moduli, respectively. ρ, C_E are, respectively, the density and specific heat at constant strain; a, b are, respectively, coefficients describing the measure of thermoelastic diffusion effects and of diffusion effects, and T_0 is the reference temperature assumed to be such that $|T/T_0| \ll 1$. $T(x_1, x_2, x_3, t)$

is the temperature change, and C is the concentration. $\sigma_{ij}(=\sigma_{ji})$, $K_{ij}(=K_{ji})$, $e_{ij} = (u_{i,j}+u_{j,i})/2$ are components of stress, thermal conductivity, and strain tensor, respectively. $\alpha^*_{ij}(=\alpha^*_{ji})$ are diffusion parameters. P, S are the chemical potential and entropy per unit mass, respectively, and $\vec{q}$, $\vec{\eta}$ denote the heat flux vector and flow of diffusion mass vector, respectively. The symbols "," and "." correspond to partial and time derivatives, respectively.

The equation of motion, entropy equation, and the equation of conservation of mass are, respectively,

$$\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i, \tag{6}$$

$$q_{i,i} + \rho T_0 \dot{S} - \rho M + P\eta_{i,i} = 0, \tag{7}$$

$$\eta_{i,i} = \dot{C} + \rho N, \tag{8}$$

where F_i is the external force per unit mass, M, N are the strengths of heat and mass diffusion source per unit mass, and u_i is the displacement vector.

Using Eqs. 1–5 in Eqs. 6–8 without body, heat, and diffusive mass forces, we obtain the equations of motion,

$$c_{ijkm}e_{km,j} + a_{ij}T_{,j} + b_{ij}C_{,j} = \rho \ddot{u}_i, \tag{9}$$

the equation of heat conduction,

$$\rho C_E \dot{T} + aT_0\dot{C} - a_{ij}\dot{e}_{ij}T_0 = K_{ij}T_{,ij}, \tag{10}$$

and the equation of mass diffusion,

$$-\alpha^*_{ij}b_{km}e_{km,ij} - \alpha^*_{ij}bC_{,ij} + \alpha^*_{ij}aT_{,ij} = -\dot{C}. \tag{11}$$

Applying the transformation,

$$x_1' = x_1\cos\phi + x_2\sin\phi, \quad x_2' = -x_1\sin\phi + x_2\cos\phi, \quad x_3' = x_3, \tag{12}$$

where ϕ is the angle of rotation in the x_1–x_2 plane, in Eqs. 9–11, the basic equations for a homogeneous, TI, coupled thermodifffusive elastic solid are

$$\begin{aligned} &c_{11}u_{1,11} + c_{12}u_{2,21} + c_{13}u_{3,31} + c_{66}(u_{1,22} + u_{2,12}) \\ &\quad + c_{44}(u_{1,33} + u_{3,13}) - a_1T_{,1} - b_1C_{,1} = \rho\ddot{u}_1, \end{aligned} \tag{13}$$

$$\begin{aligned} &c_{66}(u_{1,21} + u_{2,11}) + c_{12}u_{1,12} + c_{11}u_{2,22} + c_{44}u_{2,33} + (c_{13} \\ &\quad + c_{44})u_{3,32} - a_1T_{,2} - b_1C_{,2} = \rho\ddot{u}_2, \end{aligned} \tag{14}$$

$$\begin{aligned} &(c_{13} + c_{44})(u_{1,13} + u_{2,23}) + c_{44}(u_{3,11} + u_{3,22}) \\ &\quad + c_{33}u_{3,33} - a_3T_{,3} - b_3C_{,3} = \rho\ddot{u}_3, \end{aligned} \tag{15}$$

$$\rho C_E\dot{T} + aT_0\dot{C} + [a_1(\dot{u}_{1,1} + \dot{u}_{2,2}) + a_3\dot{u}_{3,3}]T_0 = K_1(T_{,11} + T_{,22}) + K_3T_{,33}, \tag{16}$$

$$\alpha_1^*[b_1(u_{1,111}+u_{2,222}+u_{2,211}+u_{1,122})+b_3(u_{3,311}+u_{3,322})]$$
$$+\alpha_3^*[b_1(u_{1,133}+u_{2,233})+b_3u_{3,333}]-\alpha_1^*b(C_{,11}+C_{,22})$$
$$-\alpha_3^*bC_{,33}+\alpha_1^*a(T_{,11}+T_{,22})+\alpha_3^*aT_{,33}=-\dot{C}, \quad (17)$$

where

$$\begin{aligned} a_{ij} &= -a_i\delta_{ij}, \quad b_{ij}=-b_i\delta_{ij}, \quad \alpha_{ij}^*=\alpha_i^*\delta_{ij}, \quad K_{ij}=K_i\delta_{ij},\\ a_1 &= (c_{11}+c_{12})\alpha_1+c_{13}\alpha_3, \quad a_3=2c_{13}\alpha_1+c_{33}\alpha_3,\\ b_1 &= (c_{11}+c_{12})\alpha_{1c}+c_{13}\alpha_{3c}, \quad b_3=2c_{13}\alpha_{1c}+c_{33}\alpha_{3c},\\ c_{66} &= (c_{11}-c_{12})/2. \end{aligned} \quad (18)$$

Here α_t, α_{tc} are the coefficients of linear thermal expansion and diffusion expansion, respectively.

In the above Eqs. 13–17, we use the contracting subscript notations $1 \to 11, 2 \to 22, 3 \to 33, 4 \to 23, 5 \to 13, 6 \to 12$ to relate c_{ijkm} to $c_{in}(i, j, k, m = 1, 2, 3$ and $l, n = 1, 2, 3, 4, 5, 6)$.

3 Formulation of the Problem

We consider a homogeneous, TI, coupled thermodiffusive elastic layer of thickness $2H$, initially at a uniform temperature T_0. The origin of the coordinate system (x_1, x_2, x_3) is taken on the middle surface of the layer. The x_1–x_2 plane is chosen to coincide with the middle surface with the x_3-axis normal to it along the thickness. The surfaces $x_3 = \pm H$ are subjected to different boundary conditions. We assume that the solutions are explicitly independent of x_2, but an implicit dependence is there so that the component u_2 of displacement is non-vanishing. Therefore, Eqs. 13–17 reduce to

$$c_{11}u_{1,11}+c_{44}u_{1,33}+(c_{13}+c_{44})u_{3,13}-a_1T_{,1}-b_1C_{,1}=\rho\ddot{u}_1, \quad (19)$$
$$c_{66}u_{2,11}+c_{44}u_{2,33}-a_1T_{,2}-b_1C_{,2}=\rho\ddot{u}_2, \quad (20)$$
$$(c_{13}+c_{44})u_{1,13}+c_{44}u_{3,11}+c_{33}u_{3,33}-a_3T_{,3}-b_3C_{,3}=\rho\ddot{u}_3, \quad (21)$$
$$\rho C_E\dot{T}+aT_0\dot{C}+[a_1\dot{u}_{1,1}+a_3\dot{u}_{3,3}]T_0=K_1T_{,11}+K_3T_{,33}, \quad (22)$$
$$\alpha_1^*(b_1u_{1,111}+b_3u_{3,311})+\alpha_3^*(b_1u_{1,133}+b_3u_{3,333})-\alpha_1^*bC_{,11}$$
$$-\alpha_3^*bC_{,33}+\alpha_1^*aT_{,11}+\alpha_3^*aT_{,33}=-\dot{C}. \quad (23)$$

We define the dimensionless quantities,

$$\begin{aligned} &x_i'=\frac{w_1^*x_i}{v_1}, \quad t'=w_1^*t, \quad u_i'=\frac{w_1^*u_i}{v_1}, \quad T'=\frac{a_1T}{\rho v_1^2}, \quad C'=\frac{b_1C}{\rho v_1^2},\\ &P'=\frac{P}{b_1}, \quad h'=\frac{v_1h}{w_1^*}, \quad \sigma_{ij}'=\frac{\sigma_{ij}}{a_1T_0}, \quad v_1^2=\frac{c_{11}}{\rho}, \quad w_1^*=\frac{\rho C_Ev_1^2}{K_1}. \end{aligned} \quad (24)$$

Here w_1^* is the characteristic frequency of the medium and v_1 is the longitudinal wave velocity in the medium.

Upon introducing the quantities in Eq. 24 in Eqs. 19–24, after suppressing the primes, we obtain

$$u_{1,11} + \delta_1 u_{1,33} + \delta_2 u_{3,13} - T_{,1} - C_{,1} = \ddot{u}_1, \tag{25}$$

$$\delta_3 u_{2,11} + \delta_1 u_{2,33} = \ddot{u}_2, \tag{26}$$

$$\delta_2 u_{1,13} + \delta_1 u_{3,11} + \delta_4 u_{3,33} - p_1 T_{,3} - p_2 C_{,3} = \ddot{u}_3, \tag{27}$$

$$\dot{T} + \zeta_1 \dot{C} + \zeta_2(\dot{u}_{1,1} + p_1 \dot{u}_{3,3}) = T_{,11} + p_3 T_{,33}, \tag{28}$$

$$\begin{aligned} &q_1^* u_{1,111} + q_7^* u_{1,133} + q_2^* u_{3,333} + q_8^* u_{3,311} + q_3^* T_{,11} + q_4^* T_{,33} \\ &\quad - q_5^* C_{,11} - q_6^* C_{,33} = -\dot{C}, \end{aligned} \tag{29}$$

where

$$\delta_1 = \frac{c_{44}}{c_{11}}, \quad \delta_2 = \frac{c_{13} + c_{44}}{c_{11}}, \quad \delta_3 = \frac{c_{11} - c_{12}}{2c_{11}}, \quad \delta_4 = \frac{c_{33}}{c_{11}}, \quad p_1 = \frac{a_3}{a_1}, \quad p_2 = \frac{b_3}{b_1},$$

$$p_3 = \frac{K_3}{K_1}, \quad \zeta_1 = \frac{aT_0 v_1^2 a_1}{w_1^* K_1 b_1}, \quad \zeta_2 = \frac{a_1^2 T_0}{\rho K_1 w_1^*}, \quad q_1^* = \frac{\alpha_1^* w_1^* b_1^2}{\rho v_1^4}, \quad q_2^* = \frac{\alpha_3^* w_1^* b_1 b_3}{\rho v_1^4},$$

$$q_3^* = \frac{\alpha_1^* w_1^* b_1 a}{a_1 v_1^2}, \quad q_4^* = \frac{\alpha_3^* w_1^* b_1 a}{a_1 v_1^2}, \quad q_5^* = \frac{\alpha_1^* w_1^* b}{v_1^2}, \quad q_6^* = \frac{\alpha_3^* w_1^* b}{v_1^2},$$

$$q_7^* = \frac{\alpha_3^* w_1^* b_1^2}{\rho v_1^4}, \quad q_8^* = \frac{\alpha_1^* w_1^* b_1 b_3}{\rho v_1^4}.$$

4 Solution of the Problem

We assume solutions of the form,

$$(u_1, u_2, u_3, T, C) = (1, V, W, S, R) \exp[\iota\xi(x_1 \sin\theta + m x_3 - ct)], \tag{30}$$

where $c = \frac{\omega}{\xi}$ is the dimensionless phase velocity, ω is the frequency, and ξ is the wave number. Here θ is the angle of inclination of the wave normal with the axis of symmetry (x_3-axis); m is still an unknown parameter. $1, V, W, S, R$ are, respectively, the amplitude ratios of displacements u_1, u_2, u_3, temperature change T, and concentration C with respect to u_1.

Upon using solutions of Eq. 30 in Eqs. 25–29, we obtain

$$\xi^2(s^2 + m^2\delta_1 - c^2) + \xi^2\delta_2 smW + \iota\xi sS + \iota\xi sR = 0, \tag{31}$$

$$(\delta_3 s^2 + \delta_1 m^2 - c^2)V = 0, \tag{32}$$

$$\xi^2 sm\delta_2 + \xi^2(\delta_1 s^2 + \delta_4 m^2 - c^2)W + \iota p_1 \xi mS + \iota p_2 \xi mR = 0, \tag{33}$$

$$\omega\xi s\zeta_2 + p_1\omega\xi m\zeta_2 W + (\xi^2 s^2 + p_3\xi^2 m^2 - \iota\omega)S - \iota\omega\zeta_1 R = 0, \tag{34}$$

$$\begin{aligned} &\iota\xi^2 s(q_1^* s^2 + q_7^* m^2) + \iota\xi^2 m(q_8^* s^2 + q_2^* m^2)W + \xi(q_3^* s^2 + q_4^* m^2)S \\ &\quad - \xi(q_5^* s^2 + q_6^* m^2)R = -\iota cR, \end{aligned} \tag{35}$$

where $s = \sin\theta$.

Equation 32 corresponds to the purely quasi-transverse wave mode (QSH) mode that decouples from the rest of the motion and is not affected by thermal and diffusion vibrations. The system of Eqs. 31 and 33–35 have a non-trivial solution if the determinant of the coefficients $[1, W, S, R]^{\mathrm{Tr}}$ vanishes, which yields the following polynomial characteristic equation,

$$m^8 + A^* m^6 + B^* m^4 + C^* m^2 + D^* = 0. \tag{36}$$

The coefficients A^*, B^*, C^*, D^* are given in Appendix A. The characteristic Eq. 36 is biquadratic in m^2 and hence possesses four roots m_p^2, $p = 1, 2, 3, 4$. Corresponding to these four roots, there exists three type of quasi-longitudinal waves and one quasi-transverse wave. The formal expressions for displacements, temperature change, and concentration can be written as

$$u_1 = \sum_{p=1}^{4} (A_p \cos \xi m_p x_3 + B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{37}$$

$$u_3 = \sum_{p=1}^{4} n_{1p}(A_p \cos \xi m_p x_3 + B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{38}$$

$$T = \sum_{p=1}^{4} n_{2p}(A_p \cos \xi m_p x_3 + B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{39}$$

$$C = \sum_{p=1}^{4} n_{3p}(A_p \cos \xi m_p x_3 + B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{40}$$

where A_p, B_p, $p = 1, 2, 3, 4$ are arbitrary constants. The coupling constants n_{1p}, n_{2p}, n_{3p}, $p = 1, 2, 3, 4$ are given in Appendix B.

5 Boundary Conditions

The dimensionless boundary conditions at the interface $x_3 = \pm H$ of the layer are given by

(i) Mechanical conditions (stress-free surface)

$$\begin{aligned} \sigma_{33} &= (\delta_2 - \delta_1)u_{1,1} + \delta_4 u_{3,3} - (p_1 T + p_2 C) = 0, \\ \sigma_{31} &= \delta_1(u_{3,1} + u_{1,3}) = 0, \end{aligned} \tag{41}$$

(ii) Thermal conditions

$$T_{,3} + hT = 0, \tag{42}$$

where h is the surface heat transfer coefficient. Here $h \to 0$ corresponds to thermally insulated boundaries and $h \to \infty$ refers to isothermal surfaces.

(iii) Chemical potential

$$P = u_{1,1} + p_2 u_{3,3} - n_2 C + n_1 T = 0, \tag{43}$$

where

$$n_1 = \frac{ac_{11}}{a_1 b_1}, \quad n_2 = \frac{bc_{11}}{b_1^2}.$$

6 Derivations of the Secular Equations

Substituting the values of u_1, u_3, T, and C from Eqs. 37–40 in the boundary conditions of Eqs. 41–43 at the surface $x_3 = \pm H$, we obtain a system of eight simultaneous equations, and for a non-trivial solution of a system of equations, the determinant of the coefficients of amplitudes vanishes. This, after lengthy algebraic reductions, leads to the secular equations for the plate with stress-free thermally insulated and isothermal boundaries as

$$R_1^* \left[\frac{T_2}{T_3}\right]^{\pm} + R_2^* + R_3^* \left[\frac{T_4}{T_3}\right]^{\pm} + R_4^* \left[\frac{T_2}{T_1}\right]^{\pm} + R_5^* \left[\frac{T_2 T_4}{T_1 T_3}\right]^{\pm} + R_6^* \left[\frac{T_4}{T_1}\right]^{\pm} = 0, \tag{44}$$

for stress-free insulated boundaries $h \to 0$ of the layer,

$$P_1^* \left[\frac{T_2}{T_3}\right]^{\mp} + P_2^* + P_3^* \left[\frac{T_4}{T_3}\right]^{\mp} + P_4^* \left[\frac{T_2}{T_1}\right]^{\mp} + P_5^* \left[\frac{T_2 T_4}{T_1 T_3}\right]^{\mp} + P_6^* \left[\frac{T_4}{T_1}\right]^{\mp} = 0, \tag{45}$$

for stress-free isothermal boundaries $h \to \infty$ of the layer, where

$$T_p = \tan \xi m_p H (p = 1, 2, 3, 4)$$

and

$$\begin{aligned}
R_1^* &= m_1 m_2 [(n_{11} n_{22} - n_{21} n_{12})(P_3 n_{14} - P_4 n_{13})],\\
R_2^* &= m_1 m_3 [(n_{11} n_{23} - n_{21} n_{13})(P_4 n_{12} - P_2 n_{14})],\\
R_3^* &= m_1 m_4 [(n_{11} n_{24} - n_{21} n_{14})(P_2 n_{13} - P_3 n_{12})],\\
R_4^* &= m_2 m_3 [(n_{12} n_{23} - n_{22} n_{13})(P_1 n_{14} - P_4 n_{11})],\\
R_5^* &= m_2 m_4 [(n_{22} n_{14} - n_{12} n_{24})(P_1 n_{13} - P_3 n_{11})],\\
R_6^* &= m_3 m_4 [(n_{13} n_{24} - n_{23} n_{14})(P_1 n_{12} - P_2 n_{11})],\\
P_1^* &= m_1 m_2 [(n_{12} - n_{11})(P_3 n_{24} - P_4 n_{23})],\\
P_2^* &= m_1 m_3 [(n_{11} - n_{13})(P_2 n_{24} - P_4 n_{22})],
\end{aligned}$$

$$P_3^* = m_1 m_4[(n_{14} - n_{11})(P_2 n_{23} - P_3 n_{22})],$$
$$P_4^* = m_2 m_3[(n_{13} - n_{12})(P_1 n_{24} - P_4 n_{21})],$$
$$P_5^* = m_2 m_4[(n_{12} - n_{14})(P_1 n_{23} - P_3 n_{21})],$$
$$P_6^* = m_3 m_4[(n_{14} - n_{13})(P_1 n_{22} - P_2 n_{21})],$$
$$P_p = \iota\xi(\delta_2 - \delta_1) - p_1 n_{2p} - p_2 n_{3p}, \quad p = 1, 2, 3, 4.$$

Here, superscript +1 refers to skew symmetric and −1 refers to symmetric modes of wave propagation.

7 Non-heat Conducting Solids

If the solids are not capable of conducting heat ($K_1, K_2 \to 0$), then Eq. 22 gives

$$T = -\frac{T_0}{\rho C_E}(a_1 u_{1,1} + a_3 u_{3,3}) - \frac{a T_0}{\rho C_E} C. \tag{46}$$

Using Eq. 46 in Eqs. 19–21 and 23 and with the aid of Eq. 24, we obtain

$$\varepsilon_h u_{1,11} + \delta_1 u_{1,33} + \varepsilon_k u_{3,13} + \varepsilon_l C_{,1} = \ddot{u}_1, \tag{47}$$
$$\delta_3 u_{2,11} + \delta_1 u_{2,33} = \ddot{u}_2, \tag{48}$$
$$\varepsilon_k u_{1,13} + \delta_1 u_{3,11} + \varepsilon_m u_{3,33} + \varepsilon_n C_{,3} = \ddot{u}_3, \tag{49}$$
$$h_1' u_{1,111} + h_2' u_{1,133} + h_3' u_{3,311} + h_4' u_{3,333} + h_5' C_{,11} + h_6' C_{,33} + \dot{C} = 0, \tag{50}$$

where

$$\varepsilon_h = 1 + \frac{a_1^2 T_0}{\rho^2 C_E v_1^2}, \quad \varepsilon_k = \delta_2 + \frac{a_1 a_3 T_0}{\rho^2 C_E v_1^2}, \quad \varepsilon_l = \frac{a a_1 T_0}{\rho C_E b_1} - 1,$$

$$\varepsilon_m = \delta_4 + \frac{a_3^2 T_0}{\rho^2 C_E v_1^2}, \varepsilon_n = \frac{a a_3 T_0}{\rho C_E b_1} - 1, \quad h_1' = q_1^* - \frac{\alpha_1^* a a_1 b_1 T_0}{\rho^2 C_E v_1^4},$$
$$h_2' = q_7^* - \frac{\alpha_3^* a a_1 b_1 T_0}{\rho^2 C_E v_1^4}, \quad h_3' = q_8^* - \frac{\alpha_1^* a a_3 b_1 T_0}{\rho^2 C_E v_1^4}, \quad h_4' = q_2^* - \frac{\alpha_3^* a a_3 b_1 T_0}{\rho^2 C_E v_1^4},$$

$$h_5' = -\left(q_5^* + \frac{\alpha_1^* a^2 w_1^* T_0}{\rho^2 C_E v_1^2}\right), \quad h_5' = -\left(q_6^* + \frac{\alpha_3^* a^2 w_1^* T_0}{\rho^2 C_E v_1^2}\right).$$

The frequency equation corresponding to this case is

$$W_1^*[T_2]^{\mp} + W_2^*[T_1]^{\mp} + W_3^*[\frac{T_3}{T_1 T_2}]^{\mp} + X_1^*[\frac{T_3}{T_1}]^{\mp} + X_2^*[\frac{T_3}{T_2}]^{\mp} + X_3^* = 0, \tag{51}$$

where

$$\begin{aligned}
W_1^* &= m_1 n_{11}(Y_2 n_{13} - Y_3 n_{12}), \quad W_2^* = m_2 n_{12}(Y_3 n_{11} - Y_1 n_{13}),\\
W_3^* &= m_3 n_{13}(Y_1 n_{12} - Y_2 n_{11}),\\
X_1^* &= H_1(n_{13} - n_{12}), \quad X_2^* = H_2(n_{11} - n_{13}), \quad X_3^* = H_3(n_{12} - n_{13}),\\
Y_p &= \iota\xi L_1^* + L_2^* n_{3p},\\
H_p &= \iota\xi L_3^* - L_4^* n_{3p}, \quad L_1^* = \delta_2 - \delta_1 + \frac{a_1 a_3 T_0}{\rho^2 C_E v_1^2}, \quad L_2^* = p_2 + \frac{a_3 a T_0}{\rho C_E b_1},\\
L_3^* &= 1 - \frac{a_1 a T_0}{\rho C_E b_1},\\
L_4^* &= \frac{\rho v_1^2}{b_1^2}\left(b + \frac{a^2 T_0}{\rho C_E}\right), \quad p = 1, 2, 3, 4.
\end{aligned}$$

8 Amplitudes of Displacements, Temperature Change, and Concentration

In this section, the amplitudes of displacements, temperature change, and concentration for symmetric and skew symmetric cases are given below:

$$((u_1)_{\text{sym}}, (u_1)_{\text{asym}}) = \sum_{p=1}^{4} (A_p \cos \xi m_p x_3, B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{52}$$

$$((u_3)_{\text{sym}}, (u_3)_{\text{asym}}) = \sum_{p=1}^{4} n_{1p}(B_p \sin \xi m_p x_3, A_p \cos \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{53}$$

$$((T)_{\text{sym}}, (T)_{\text{asym}}) = \sum_{p=1}^{4} n_{2p}(A_p \cos \xi m_p x_3, B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)], \tag{54}$$

$$((C)_{\text{sym}}, (C)_{\text{asym}}) = \sum_{p=1}^{4} n_{3p}(A_p \cos \xi m_p x_3, B_p \sin \xi m_p x_3) \exp[\iota\xi(x_1 \sin\theta - ct)]. \tag{55}$$

The amplitudes A_p, B_p, $p = 1, 2, 3, 4$ are given in Appendix C.

9 Particular Cases

1. In the absence of a diffusion effect, i.e., if we take $b_1 = b_3 = a = b = 0$, Eqs. 44 and 45 yield the frequency equations of a TI-coupled thermoelastic solid as

$$S_1^*\left[\frac{T_1}{T_3}\right]^{\pm} - S_2^*\left[\frac{T_2}{T_3}\right]^{\pm} + S_3^* = 0, \tag{56}$$

$$Q_1^*\left[\frac{T_1}{T_3}\right]^{\mp} - Q_2^*\left[\frac{T_2}{T_3}\right]^{\mp} + Q_3^* = 0, \tag{57}$$

where

$$\begin{aligned}
S_1^* &= m_1 n'_{21}(V_2 n'_{13} - V_3 n'_{12}), \quad S_2^* = m_2 n'_{22}(V_1 n'_{13} - V_3 n'_{11}),\\
S_3^* &= m_3 n'_{23}(V_1 n'_{12} - V_2 n'_{11}),\\
Q_1^* &= m_1(V_2 n'_{23} - V_3 n'_{22}), \quad Q_2^* = m_2(V_1 n'_{23} - V_3 n'_{21}),\\
Q_3^* &= m_3(V_1 n'_{22} - V_2 n'_{21}),\\
n'_{1p} &= -\frac{f_3 r_4 m_p^3 + (f_3 r_3 - f_7 r_1) m_p}{f_6 r_4 m_p^4 + (f_5 r_4 + f_6 r_3 - f_7 r_2) m_p^2 + f_5 r_3},\\
n'_{2p} &= \frac{(f_3 r_2 - r_1 f_6) m_p^2 - r_1 f_5}{f_6 r_4 m_p^4 + (f_5 r_4 + f_6 r_3 - f_7 r_2) m_p^2 + f_5 r_3},\\
V_p &= \iota\xi(\delta_2 - \delta_1) - p_1 n'_{2p}, \quad p = 1, 2, 3.
\end{aligned}$$

The above results are similar to those obtained by Sharma et al. ([24], Eqs. 24 and 28).

2. For the isotropic case, we have

$$\begin{aligned}
&c_{11} = c_{33} = \lambda + 2\mu, \quad c_{12} = c_{13} = \lambda, \quad c_{44} = \mu, \quad a_1 = a_3 = \beta_1,\\
&b_1 = b_3 = \beta_2, \quad K_1 = K_2 = K, \quad \alpha_1^* = \alpha_3^* = D.
\end{aligned} \tag{58}$$

Consequently, Eqs. 44 and 45 become the frequency equations of an isotropic-coupled thermoelastic diffusive solid as

$$M_1^*\left[\frac{T_2}{T_3}\right]^{\pm} + M_2^* + M_3^*\left[\frac{T_4}{T_3}\right]^{\pm} + M_4^*\left[\frac{T_2}{T_1}\right]^{\pm} + M_5^*\left[\frac{T_2 T_4}{T_1 T_3}\right]^{\pm} + M_6^*\left[\frac{T_4}{T_1}\right]^{\pm} = 0, \tag{59}$$

$$N_1^*\left[\frac{T_2}{T_3}\right]^{\mp} + N_2^* + N_3^*\left[\frac{T_4}{T_3}\right]^{\mp} + N_4^*\left[\frac{T_2}{T_1}\right]^{\mp} + N_5^*\left[\frac{T_2 T_4}{T_1 T_3}\right]^{\mp} + N_6^*\left[\frac{T_4}{T_1}\right]^{\mp} = 0, \tag{60}$$

where

$$\begin{aligned}
M_1^* &= m_1 m_2[(m_{11} m_{22} - m_{21} m_{12})(H_3 m_{14} - H_4 m_{13})],\\
M_2^* &= m_1 m_3[(m_{11} m_{23} - m_{21} m_{13})(H_4 m_{12} - H_2 m_{14})],\\
M_3^* &= m_1 m_4[(m_{11} m_{24} - m_{21} m_{14})(H_2 m_{13} - H_3 m_{12})],\\
M_4^* &= m_2 m_3[(m_{12} m_{23} - m_{22} m_{13})(H_1 m_{14} - H_4 m_{11})],
\end{aligned}$$

$$
\begin{aligned}
M_5^* &= m_2 m_4[(m_{22}m_{14} - m_{12}m_{24})(H_1 m_{13} - H_3 m_{11})],\\
M_6^* &= m_3 m_4[(m_{13}m_{24} - m_{23}m_{14})(H_1 m_{12} - H_2 m_{11})],\\
N_1^* &= m_1 m_2[(m_{12} - m_{11})(H_3 m_{24} - H_4 m_{23})],\\
N_2^* &= m_1 m_3[(m_{11} - m_{13})(H_2 m_{24} - H_4 m_{22})],\\
N_3^* &= m_1 m_4[(m_{14} - m_{11})(H_2 m_{23} - H_3 m_{22})],\\
N_4^* &= m_2 m_3[(m_{13} - m_{12})(H_1 m_{24} - H_4 m_{21})],\\
N_5^* &= m_2 m_4[(m_{12} - m_{14})(H_1 m_{23} - H_3 m_{21})],\\
N_6^* &= m_3 m_4[(m_{14} - m_{13})(H_1 n_{22} - H_2 n_{21})],\\
H_p &= \iota\xi(\delta_2 - \delta_1) - m_{2p} - m_{3p}, \quad p = 1, 2, 3, 4.
\end{aligned}
$$

The coupling constants m_{qp} can be obtained from $n_{qp}, q = 1, 2, 3$ and $p = 1, 2, 3, 4$ by using the quantities given in Eq. 58.

2.1 Sub-case

In the absence of a diffusion effect, i.e., $a = b = \beta_2 = 0$, Eqs. 59 and 60 reduce to the frequency equations of isotropic-coupled thermoelastic solid as

$$U_1^* \left[\frac{T_1}{T_3}\right]^{\pm} - U_2^* \left[\frac{T_2}{T_3}\right]^{\pm} + U_3^* = 0, \tag{61}$$

$$V_1^* \left[\frac{T_1}{T_3}\right]^{\mp} - V_2^* \left[\frac{T_2}{T_3}\right]^{\mp} + V_3^* = 0, \tag{62}$$

where

$$
\begin{aligned}
U_1^* &= m_1 m_{21}(L_2 m'_{13} - L_3 m'_{12}), \quad U_2^* = m_2 m'_{22}(L_1 m'_{13} - L_3 m'_{11}),\\
U_3^* &= m_3 m'_{23}(L_1 m'_{12} - L_2 m'_{11}),\\
V_1^* &= m_1(L_2 m'_{23} - L_3 m'_{22}), \quad V_2^* = m_2(L_1 m'_{23} - L_3 m'_{21}),\\
V_3^* &= m_3(L_1 m'_{22} - L_2 m'_{21}),\\
L_p &= \iota\xi(\delta_2 - \delta_1) - m'_{2p}, \quad p = 1, 2, 3.
\end{aligned}
$$

The coupling constants m'_{qp} can be obtained from $n'_{qp}, q = 1, 2$ and $p = 1, 2, 3$ using the quantities given in Eq. 58.

The above results are similar to those obtained by Sharma et al. ([24], Eq. 34).

10 Numerical Results and Discussion

The material chosen for the purpose of numerical calculation is copper which is a TI material. The physical data for a single crystal of copper material are given below:

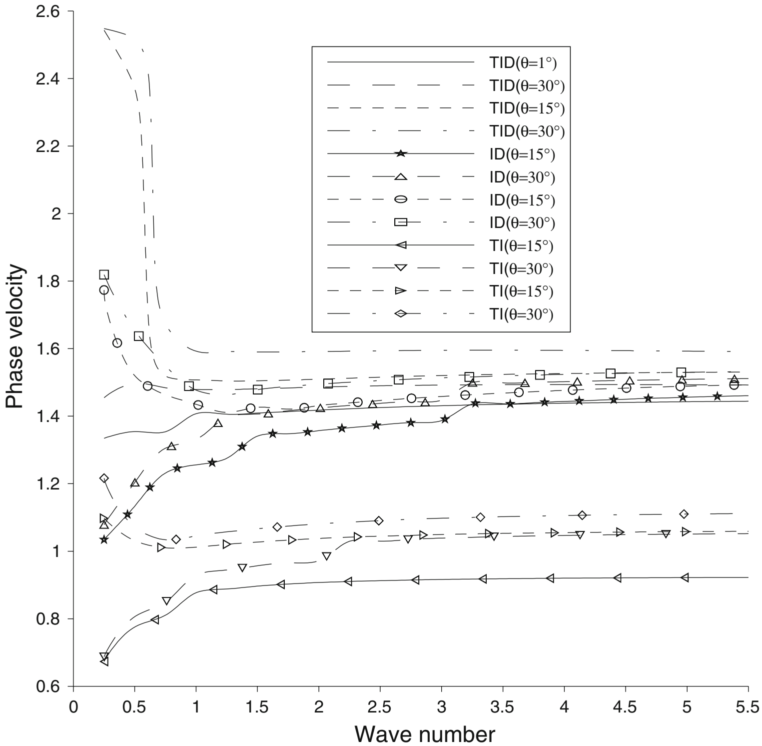

Fig. 1 Variations of phase velocity with respect to wave number (symmetric)

$$
\begin{aligned}
c_{11} &= 18.78 \times 10^{10}\ \mathrm{kg \cdot m^{-1} \cdot s^2}, \quad c_{12} = 8.76 \times 10^{10}\ \mathrm{kg \cdot m^{-1} \cdot s^2},\\
c_{13} &= 8.0 \times 10^{10}\ \mathrm{kg \cdot m^{-1} \cdot s^2},\\
c_{33} &= 18.2 \times 10^{10}\ \mathrm{kg \cdot m^{-1} \cdot s^2}, \quad c_{44} = 5.06 \times 10^{10}\ \mathrm{kg \cdot m^{-1} \cdot s^2},\\
T_0 &= 0.293 \times 10^3\ \mathrm{K}\\
C_E &= 0.6331 \times 10^3\ \mathrm{J \cdot\ kg^{-1} \cdot K^{-1}}, \alpha_1 = 2.98 \times 10^{-5}\ \mathrm{K^{-1}},\\
\alpha_3 &= 2.4 \times 10^{-5}\ \mathrm{K^{-1}},\\
\alpha_{1c} &= 2.1 \times 10^{-4}\ \mathrm{m^3 \cdot\ kg^{-1}}, \alpha_{3c} = 2.5 \times 10^{-4}\ \mathrm{m^3 \cdot\ kg^{-1}},\\
a &= 2.4 \times 10^4\ \mathrm{m^2 \cdot s^{-2} \cdot K^{-1}},\\
b &= 13.2 \times 10^5\ \mathrm{kg \cdot m^5 \cdot s^{-2}}, \alpha_1{}^* = 0.95 \times 10^{-8}\ \mathrm{kg \cdot m^{-3} \cdot s},\\
\alpha_3{}^* &= 0.90 \times 10^{-8}\ \mathrm{kg \cdot m^{-3} \cdot s},\\
\rho &= 8.954 \times 10^3\ \mathrm{kg \cdot m^{-3}}, \quad K_1 = 0.433 \times 10^3\ \mathrm{W \cdot m^{-1} \cdot K^{-1}},\\
K_3 &= 0.450 \times 10^3\ \mathrm{W \cdot m^{-1} \cdot K^{-1}}.
\end{aligned}
$$

10.1 Phase Velocity and Attenuation Coefficient

The variations of phase velocity for thermoelastic diffusive waves for first and second modes have been plotted in Figs. 1 and 2 for various modes of propagation with respect

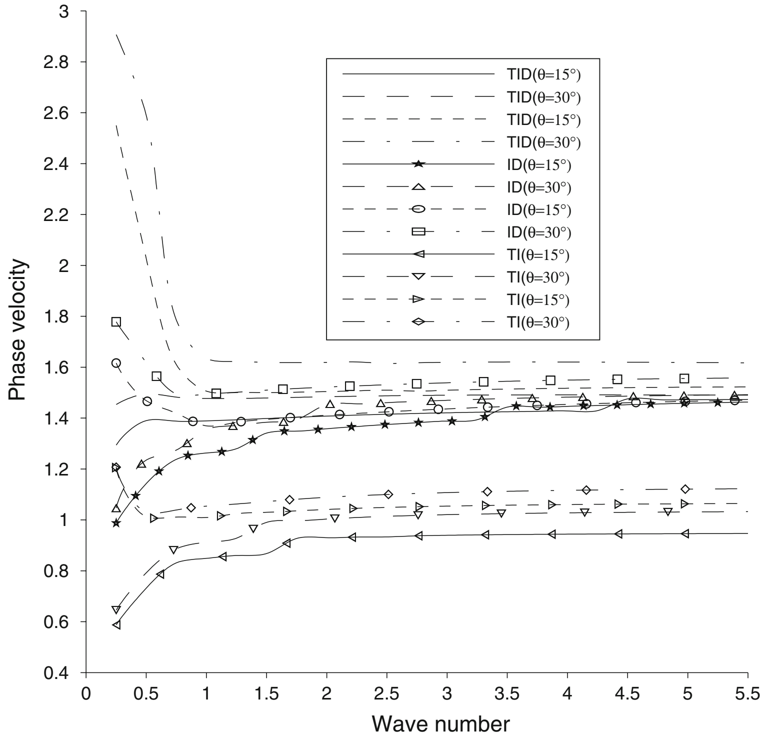

Fig. 2 Variations of phase velocity with respect to wave number (skew-symmetric)

to wave number for skew symmetric and symmetric modes. In Figs. 1, 2, 3 and 4, the solid line and big dashed line correspond to the first mode for the case of a TI thermoelastic diffusive solid (TID) for $\theta = 15°$ and $\theta = 30°$, respectively. Similarly, the small dashed line and line with dots correspond to the second mode for the case of TID for $\theta = 15°$ and $\theta = 30°$, respectively. The star, triangle, circle, and square symbols on these four lines correspond to an isotropic thermoelastic (ID) diffusive solid for the first and second modes, respectively. Similarly, the left triangle, lower triangle, right triangle, and diamond symbols correspond to a TI thermoelastic solid for the first and second modes, respectively.

From Fig. 1, it is noticed that corresponding to the case of TID, the values of the phase velocity increase slowly before remaining constant for the first mode and for $\theta = 15°$. For $\theta = 30°$, the phase velocity decreases slowly before showing a constant behavior. The values of the phase velocity increase with the variation of angle from a lower value to a higher value. On the other hand, initially, values of the phase velocity decrease sharply but finally remain constant for the second mode and for both angles. If we compare first and second modes, we find that the values of phase velocity corresponding to the second mode are higher than those of the first mode. A similar trend is noticed for the case of ID, except that the values of the phase velocity increase at first and then remain constant for the first mode and for $\theta = 30°$. The values of

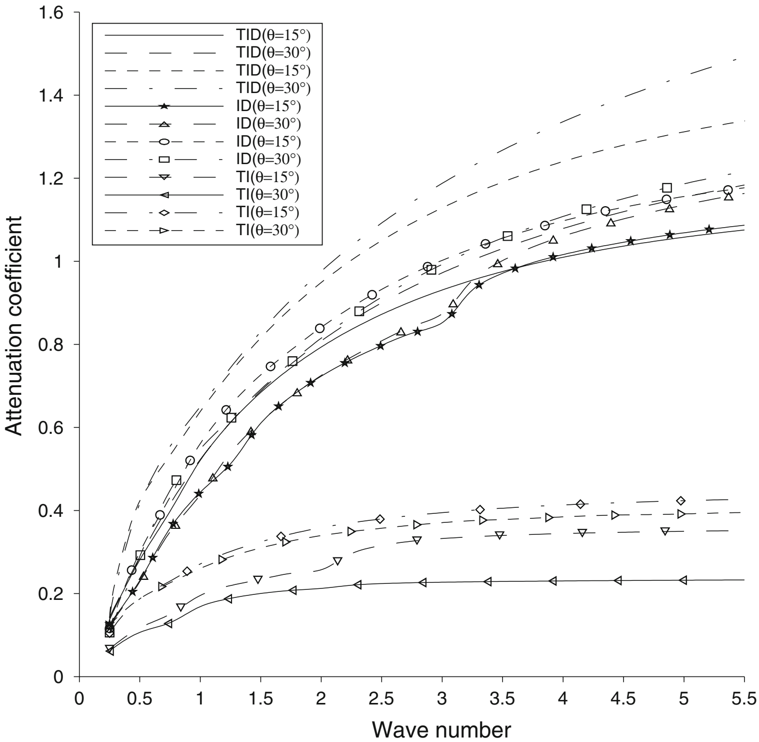

Fig. 3 Variations of attenuation coefficient with respect to wave number (symmetric)

the phase velocity for the case of the first mode for TID are larger in comparison to ID with the wave number, and as the wave number increases, the reverse behavior occurs, whereas for the cases of TID and TI, we find that there is a larger increase for the values of TID in comparison to TI. Thus, an appreciable diffusion effect as compared to an anisotropy effect is noticed. Figure 2 shows similar behavior for the skew-symmetric mode, but the magnitudes are different as compared to the symmetric mode.

The corresponding attenuation coefficients are shown in Figs. 3 and 4. The values of the attenuation coefficient increase for all three cases of TI, ID, and TID. The values of the attenuation coefficient for TID are larger in comparison to TI and ID. As the angle increases from $\theta = 15°$ to $\theta = 30°$, there is a large increase in the values of the attenuation coefficient.

10.2 Amplitudes

The variations of amplitudes of displacements (u_1, u_3), temperature change (T), and concentration (C) with respect to the thickness (H) of the layer have been computed and are shown in Figs. 5, 6, 7, 8, 9, 10, 11 and 12 for different angles. The solid and dashed lines correspond to TID for $\theta = 15°$ and $\theta = 30°$, respectively. The star and

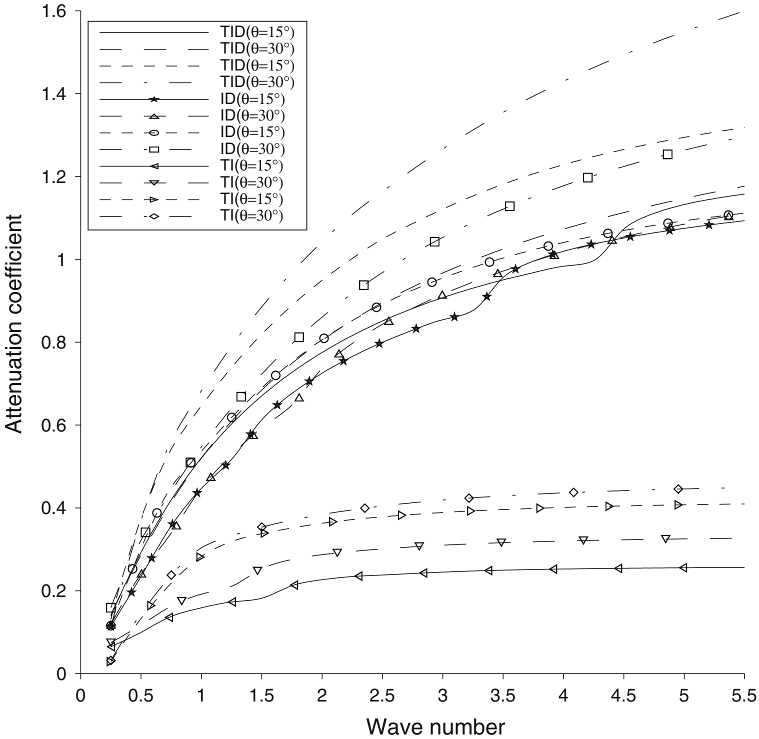

Fig. 4 Variations of attenuation coefficient with respect to wave number (skew-symmetric)

circle symbols on these two lines correspond to ID. Similarly, the triangle and square symbols on these lines correspond to TI.

Figure 5 depicts that the values of the amplitude of the horizontal displacement (u_1) decrease monotonically for TID, and the decrease is less as the angle moves from $\theta = 15°$ to $\theta = 30°$. On the contrary, the displacement (u_1) increases monotonically for the case of ID and the increase is larger as the angle is varied from a higher value to a lower value. Corresponding to TID and TI, we notice that the displacement (u_1) decreases for both cases but the decrease in the values is larger for the case of TID compared to TI. Similarly, for the skew-symmetric mode, the displacement (u_1) decreases for TID but the decrease in the values is larger with the variation of the angle from $\theta = 15°$ to $\theta = 30°$, in contrary to the symmetric mode. On the other hand, u_1 increases for the case of ID and the increase is less as the angle moves from $\theta = 15°$ to $\theta = 30°$. For the cases of TID and TI, the trend is similar as for the case of the symmetric mode. The values of the amplitude of the vertical displacement (u_3) are oscillatory in nature. The displacement (u_3) vibrates more for TID than for the cases of ID and TI. The values of the amplitude of the vertical displacement (u_3) are multiplied by 10^6 to depict anisotropy and diffusion effects.

In Fig. 9, the values of the temperature change (T) increase monotonically corresponding to TID and decrease monotonically corresponding to ID for the symmet-

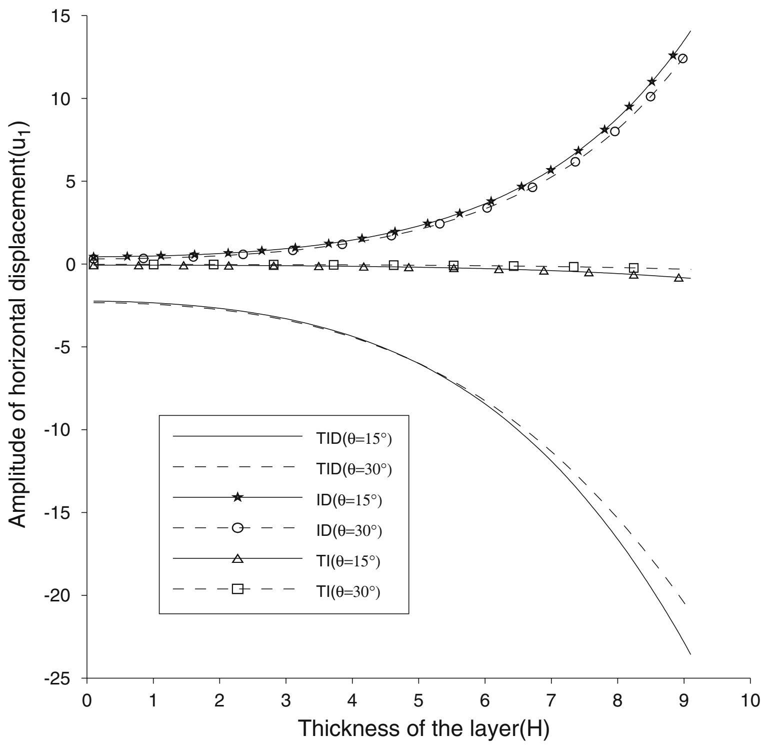

Fig. 5 Variations of amplitude of horizontal surface displacement with respect to thickness of layer H (symmetric)

ric mode; but for the skew-symmetric mode, the temperature change (T) decreases monotonically in both cases. The values of the temperature change (T) are magnified by 10^3 for the case of TI. A similar trend is noticed corresponding to TID and TI, but for skew-symmetric mode, a decrease in the temperature change (T) is larger than for the case of TID as compared to TI.

Figure 11 shows that for the case of TID, the values of concentration (C) increase sharply for $\theta = 15°$ and decreases slowly for $\theta = 30°$. On the other hand, the concentration (C) increases for the case of ID for both angles. A similar trend is shown in Fig. 12 corresponding to the case of ID. For the case of TID, the concentration (C) decreases for both angles up to a certain limit but after that, it increases for $\theta = 30°$.

11 Conclusions

The propagation of plane harmonic thermoelastic diffusive waves in a homogeneous, transversely isotropic, thin elastic layer of finite width is studied, in the context of the coupled theory of thermoelastic diffusion. It is shown that there are three quasi-longitudinal waves, namely, QED-mode, QMD-mode, and QTD-mode, in addition to the two quasi-transverse waves (QSV-mode and QSH-mode). The quasi-transverse

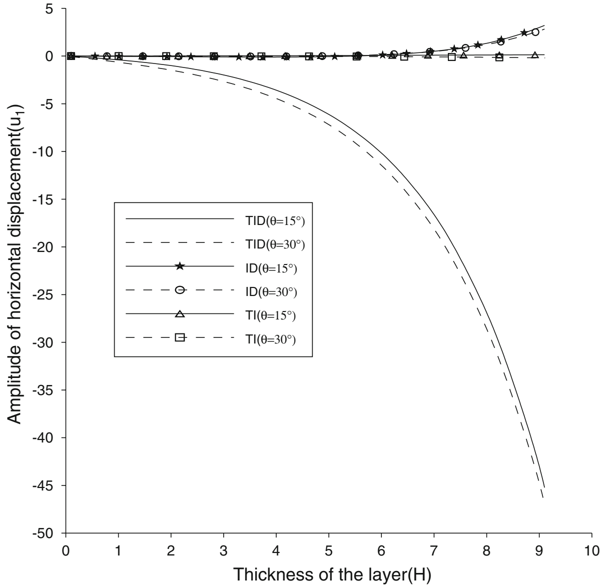

Fig. 6 Variations of amplitude of horizontal surface displacement with respect to thickness of layer H (skew-symmetric)

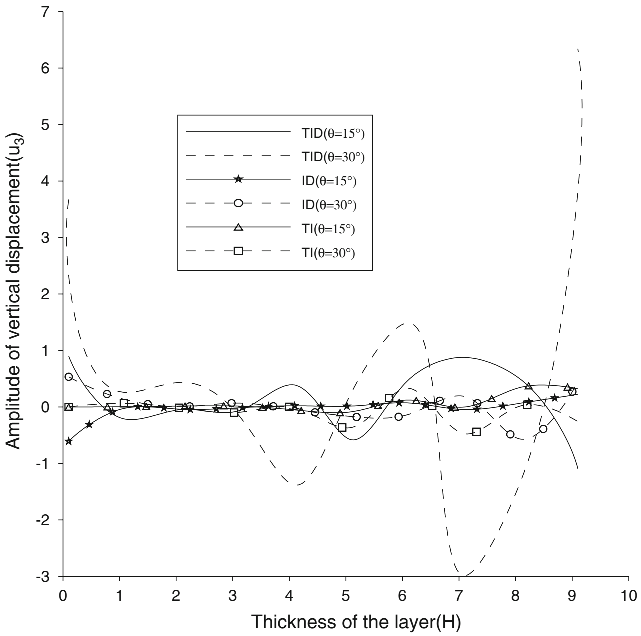

Fig. 7 Variations of amplitude of vertical surface displacement with respect to thickness of layer H (symmetric)

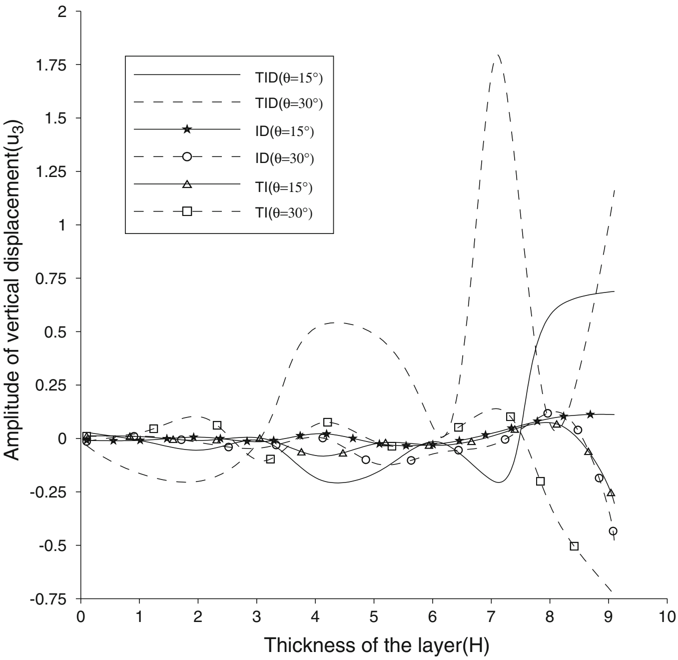

Fig. 8 Variations of amplitude of vertical surface displacement with respect to thickness of layer H (skew-symmetric)

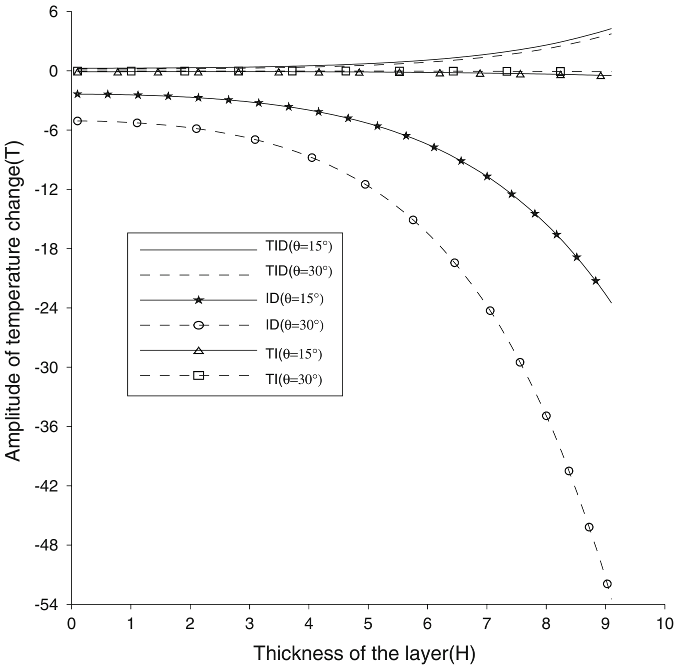

Fig. 9 Variations of amplitude of temperature change with respect to thickness of layer H (symmetric)

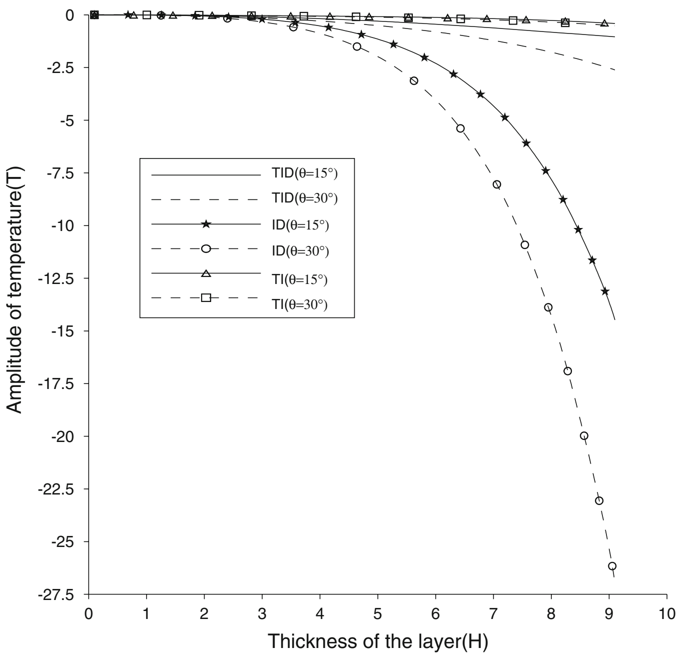

Fig. 10 Variations of amplitude of temperature change with respect to thickness of layer H (skew-symmetric)

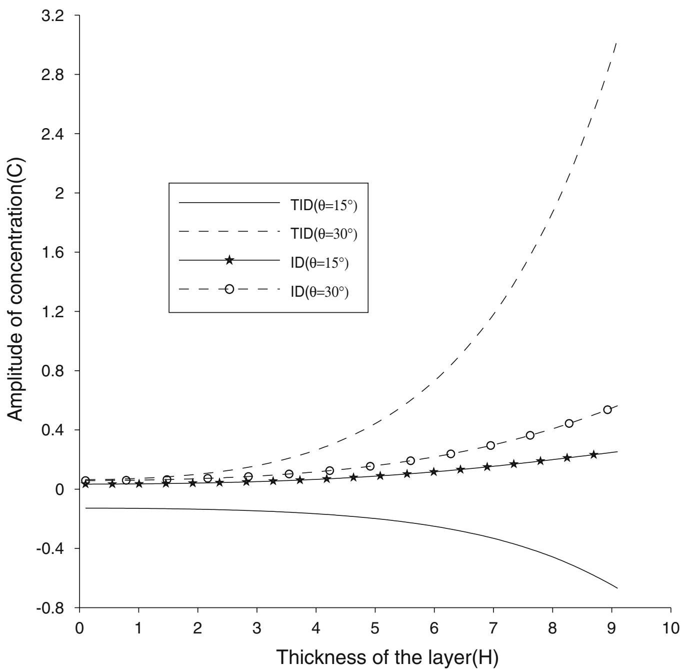

Fig. 11 Variations of amplitude of concentration with respect to thickness of layer H (symmetric)

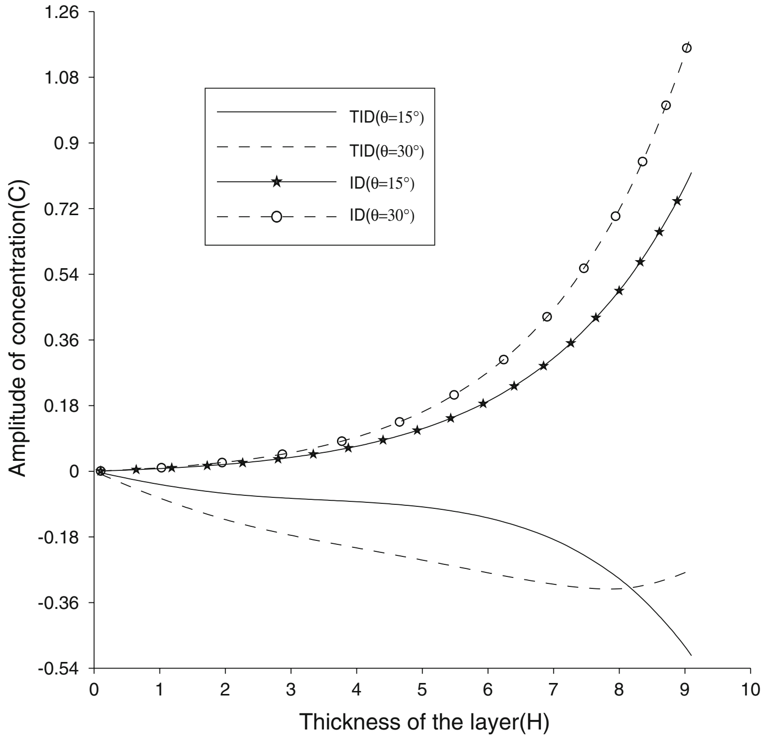

Fig. 12 Variations of amplitude of concentration with respect to thickness of layer H (skew-symmetric)

waves (QSH-mode), which are not affected by thermal and diffusion vibrations, get decoupled from the rest of the motion of wave propagation. Some special cases of the frequency equation are also discussed. Anisotropy and diffusion effects on the phase velocity, attenuation coefficient, and amplitudes of wave propagation are shown graphically, and the results are compared with existing results.

It is shown that the values of the phase velocity for the second mode show a larger increase than those of the first mode, and the values increase with an increase in angle. Due to anisotropy and diffusion effects, the increase in the values of the phase velocity and attenuation coefficient is smaller. Appreciable diffusion and anisotropy effects on amplitudes are observed.

Appendix A

Coefficients of Eq. 36

$$A^* = \frac{f_1 g_1 + f_2 g_2 - f_3 g_3 - r_7 g_7}{f_2 g_1}, \quad B^* = \frac{f_1 g_2 + f_2 g_4 - f_3 g_5 + r_1 g_6 - r_6 g_7 - r_7 g_{11}}{f_2 g_1},$$

$$C^* = \frac{f_1 g_4 + f_2 g_8 - f_3 g_9 + r_1 g_{10} - r_6 g_{11} + r_7 g_{13}}{f_2 g_1}, \quad D^* = \frac{f_1 g_8 - r_1 g_{12} + r_6 g_{13}}{f_2 g_1},$$

$$\begin{aligned}
g_1 &= f_6 l_3 - r_8 l_7, \quad g_2 = f_5 l_3 + f_6 l_2 - r_2 l_5 - r_9 l_7 + r_8 l_6, \quad g_3 = f_3 l_3 - r_8 j_2,\\
g_4 &= f_5 l_2 + f_6 l_1 - r_2 l_4 + r_9 l_6,\\
g_5 &= f_3 l_2 - r_2 l_9 + r_8 j_1 - r_9 j_2, \quad g_6 = f_3 l_5 - f_6 l_9 + r_8 j_3,\\
g_7 &= f_6 j_2 - f_3 l_7, \quad g_8 = f_5 l_1,\\
g_9 &= f_3 l_1 - r_2 l_8 + r_9 j_1, \quad g_{10} = f_3 l_4 - f_5 l_9 - f_6 l_8 + r_9 j_3,\\
g_{11} &= f_3 l_6 + f_5 j_2 - f_6 j_1 + r_2 j_3,\\
g_{12} &= f_5 l_8, \quad g_{13} = f_5 j_1, \quad f_1 = \xi^2 (s^2 - c^2), \quad f_2 = \delta_1 \xi^2, \quad f_3 = \delta_2 s \xi^2,\\
f_4 &= \iota \xi s, \quad f_5 = \xi^2 (\delta_1 s^2 - c^2),\\
f_6 &= \delta_4 c^2, \ f_7 = \iota \xi p_1, \ f_8 = \iota \xi p_2, \quad r_1 = \xi \omega s \zeta_2, \quad r_2 = \xi \omega \zeta_2 p_1, \ r_3 = \xi^2 s^2 - \iota \omega,\\
r_4 &= p_3 \xi^2, \quad r_5 = -\iota \omega \zeta_1,\\
r_6 &= \iota q_1^* \xi^2 s^3, \quad r_7 = \iota q_7^* \xi^2 s, \quad r_8 = \iota q_2^* \xi^2, \quad r_9 = \iota q_8^* \xi^2 s^2, \quad h_1 = q_3^* \xi s^2,\\
h_2 &= q_4^* \xi, \quad h_3 = -q_5^* \xi s^2 + \iota c, \quad h_4 = -q_6^* \xi,\\
l_1 &= r_3 h_3 - r_5 h_1, \quad l_2 = h_4 r_3 + r_4 h_3 - h_2 r_5, \quad l_3 = r_4 h_4, \quad l_4 = f_7 h_3 - f_8 h_1,\\
l_5 &= f_7 h_4 - h_2 f_8, \quad l_6 = f_7 r_5 - f_8 r_3,\\
l_7 &= r_4 f_8, \quad l_8 = f_4 (h_3 - h_1), \quad l_9 = (h_4 - h_2) f_4,\\
j_1 &= f_4 (r_5 - r_3), \quad j_2 = r_4 f_4, \quad j_3 = f_4 (f_8 - f_7).
\end{aligned}$$

Appendix B

Coupling Constants of Eqs. 37–40

$$\begin{aligned}
I_1 &= r_2 h_4 - r_8 r_5, \quad I_2 = f_6 h_4 - f_8 r_8, \quad I_3 = f_6 r_5 - f_8 r_2, \quad I_4 = r_2 h_3 - r_5 r_9,\\
I_5 &= f_6 h_3 + f_5 h_4 - f_8 r_9,\\
I_6 &= r_6 r_5 - r_1 h_3, \quad I_7 = f_6 r_7 - f_3 r_8, \quad I_8 = h_2 r_2 - r_8 r_3 - r_9 r_4, \ I_9 = f_6 h_2 - f_7 r_8,\\
I_{10} &= f_6 r_3 + f_5 r_4 + f_7 r_2,\\
I_{11} &= r_2 h_1 - r_9 r_3, \quad I_{12} = f_6 h_1 + f_5 h_2 - f_7 r_9, \quad I_{13} = r_6 r_3 - h_1 r_1,\\
g_{14} &= f_3 l_1 - r_7 l_7, \quad g_{15} = f_3 l_2 - r_1 l_5 - r_6 l_7 + r_7 l_6,\\
g_{16} &= f_3 l_3 - r_1 l_4 + r_6 l_6, \quad g_{17} = f_3 I_1 - r_1 I_2 + r_7 I_3,\\
g_{18} &= f_3 I_4 - r_1 I_5 + r_6 I_3 + r_7 r_5 f_5, \quad g_{19} = f_5 I_6,\\
g_{20} &= r_4 I_7, \quad g_{21} = f_3 I_8 - r_1 I_9 + r_7 I_{10} + r_6 f_6 r_4,\\
g_{22} &= f_3 I_{11} - r_1 I_{12} + r_6 I_{10} + r_7 f_5 r_3, \quad g_{23} = f_5 I_{13},\\
n_{1p} &= -\frac{m_p^5 g_{14} + m_p^3 g_{15} + m_p g_{16}}{m_p^6 g_1 + m_p^4 g_2 + m_p^2 g_4 + g_8}, \quad n_{2p} = \frac{m_p^4 g_{17} + m_p^2 g_{18} + g_{19}}{m_p^6 g_1 + m_p^4 g_2 + m_p^2 g_4 + g_8},\\
n_{3p} &= -\frac{m_p^6 g_{20} + m_p^4 g_{21} + m_p^2 g_{22} + g_{23}}{m_p^6 g_1 + m_p^4 g_2 + m_p^2 g_4 + g_8}, \quad p = 1, 2, 3, 4.
\end{aligned}$$

Appendix C

Amplitudes of Eqs. 52–55

$$
\begin{aligned}
&\frac{A_1}{\mathrm{Det}(I)} = -\frac{A_2}{\mathrm{Det}(II)} = \frac{A_3}{\mathrm{Det}(III)} = -\frac{A_4}{\mathrm{Det}(IV)} = \frac{B_1}{\mathrm{Det}(V)}\\
&= -\frac{B_2}{\mathrm{Det}(VI)} = \frac{B_3}{\mathrm{Det}(VII)} = -\frac{B_4}{\mathrm{Det}(VIII)},\\
&c_p = \cos \xi m_p x_3, \quad s_p = \sin \xi m_p x_3, \quad p = 1, 2, 3, 4
\end{aligned}
$$

$$
\begin{aligned}
d_1^* &= c_1 m_2 s_2 - c_2 m_1 s_1, \quad d_2^* = c_1 m_3 s_3 - c_3 m_1 s_1, \quad d_3^* = c_1 m_4 s_4 - c_4 m_1 s_1,\\
d_4^* &= c_2 m_3 s_3 - c_3 m_2 s_2,\\
d_5^* &= c_2 m_4 s_4 - c_4 m_2 s_2, \quad d_6^* = c_3 m_4 s_4 - c_4 m_3 s_3, \quad d_7^* = c_1 m_1 s_2 - c_2 m_2 s_1,\\
d_8^* &= c_1 m_1 s_3 - c_3 m_3 s_1,\\
d_9^* &= c_1 m_1 s_4 - c_4 m_4 s_1, \quad d_{10}^* = c_2 m_2 s_3 - c_3 m_3 s_2, \quad d_{11}^* = c_2 m_2 s_4 - c_4 m_4 s_2,\\
d_{12}^* &= c_3 m_3 s_4 - c_4 m_4 s_3,\\
h_1^* &= m_1 m_2 c_1 c_2 s_3 s_4 (P_4 n_{13} - P_3 n_{14})(n_{21} n_{12} - n_{11} n_{22}),\\
h_2^* &= m_1 m_3 c_1 c_3 s_2 s_4 (P_2 n_{14} - P_4 n_{12})(n_{21} n_{13} - n_{23} n_{11}),\\
h_3^* &= m_1 m_4 c_1 c_4 s_2 s_3 (P_3 n_{12} - P_2 n_{13})(n_{21} n_{14} - n_{24} n_{11}),\\
h_4^* &= m_2 m_3 c_2 c_3 s_1 s_4 (P_4 n_{11} - P_1 n_{14})(n_{22} n_{13} - n_{12} n_{23}),\\
h_5^* &= m_2 m_4 c_2 c_4 s_1 s_3 (P_1 n_{13} - P_3 n_{11})(n_{22} n_{14} - n_{24} n_{12}),\\
h_6^* &= m_3 m_4 c_3 c_4 s_1 s_2 (P_2 n_{11} - P_1 n_{12})(n_{23} n_{14} - n_{24} n_{13}),\\
h_7^* &= -m_1 m_2 s_1 s_2 c_3 c_4 (P_4 n_{13} - P_3 n_{14})(n_{21} n_{12} - n_{11} n_{22}),\\
h_8^* &= -m_1 m_3 s_1 s_3 c_2 c_4 (P_2 n_{14} - P_4 n_{12})(n_{21} n_{13} - n_{23} n_{11}),\\
h_9^* &= -m_1 m_4 s_1 s_4 c_2 c_3 (P_3 n_{12} - P_2 n_{13})(n_{21} n_{14} - n_{24} n_{11}),\\
h_{10}^* &= -m_2 m_3 s_2 s_3 c_1 c_4 (P_4 n_{11} - P_1 n_{14})(n_{22} n_{13} - n_{12} n_{23}),\\
h_{11}^* &= -m_2 m_4 s_2 s_4 c_1 c_3 (P_1 n_{13} - P_3 n_{11})(n_{22} n_{14} - n_{24} n_{12}),\\
h_{12}^* &= -m_3 m_4 s_3 s_4 c_1 c_2 (P_2 n_{11} - P_1 n_{12})(n_{23} n_{14} - n_{24} n_{13}),\\
D_1^* &= h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^*, \; D_2^* = h_7^* + h_8^* + h_9^* + h_{10}^* + h_{11}^* + h_{12}^*,\\
\mathrm{Det}(I) &= -8 D_1^* p_2 \xi^6 [n_{13} n_{14} n_{22} m_2 s_2 d_6^* - n_{12} n_{14} n_{23} m_3 s_3 d_5^* + n_{12} n_{13} n_{24} m_4 s_4 d_4^*],\\
\mathrm{Det}(II) &= -8 D_1^* p_2 \xi^6 [n_{13} n_{14} n_{21} m_1 s_1 d_6^* - n_{11} n_{14} n_{23} m_3 s_3 d_3^* + n_{11} n_{13} n_{24} m_4 s_4 d_2^*],\\
\mathrm{Det}(III) &= -8 D_1^* p_2 \xi^6 [n_{12} n_{14} n_{21} m_1 s_1 d_5^* - n_{11} n_{14} n_{22} m_2 s_2 d_3^* + n_{11} n_{12} n_{24} m_4 s_4 d_1^*],\\
\mathrm{Det}(IV) &= -8 D_1^* p_2 \xi^6 [n_{13} n_{12} n_{21} m_1 s_1 d_4^* - n_{11} n_{13} n_{22} m_2 s_2 d_2^* + n_{12} n_{11} n_{23} m_3 s_3 d_1^*],\\
\mathrm{Det}(V) &= -8 D_2^* p_2 \xi^6 [n_{13} n_{14} n_{22} m_2 c_2 d_{12}^* - n_{12} n_{14} n_{23} m_3 c_3 d_{11}^*\\
&\qquad + n_{12} n_{13} n_{24} m_4 c_4 d_{10}^*],\\
\mathrm{Det}(VI) &= -8 D_2^* p_2 \xi^6 [n_{13} n_{14} n_{21} m_1 c_1 d_{12}^* - n_{11} n_{14} n_{23} m_3 c_3 d_9^*\\
&\qquad + n_{11} n_{13} n_{24} m_4 c_4 d_8^*],
\end{aligned}
$$

$$\mathrm{Det}(VII) = -8D_2^* p_2 \xi^6 [n_{12}n_{14}n_{21}m_1c_1d_{11}^* - n_{11}n_{14}n_{22}m_2c_2d_9^* + n_{11}n_{12}n_{24}m_4c_4d_7^*],$$
$$\mathrm{Det}(VIII) = -8D_2^* p_2 \xi^6 [n_{13}n_{12}n_{21}m_1c_1d_{10}^* - n_{11}n_{13}n_{22}m_2c_2d_8^* + n_{12}n_{11}n_{23}m_3c_3d_7^*].$$

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